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ON OPTIMAL STRATEGIES FOR A BETTING GAME

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The game n-bet is to bet n times with each bet between 0 and 1 inclusive. To win the game means to have a specified positive integer m as the final result. If the probability of winning any bet is a constant p, then the strategy $N(n,m)$ of betting 1 every time until m is obtained is optimal when $0<p\le\frac{1}{2}$, and is not always optimal when $\frac{1}{2}<p<1$.

1. Introduction

In [2] Klawe considered the following problem. Suppose in a one-person game the player can bet up to n times and the amount of each bet is between 0 and 1 inclusive as his choice. The result of any bet is known before the next bet, and the probability to win any bet is a constant p in the interval $(0,1)$. The result of the game is the sum of the winning bets minus the sum of the losing bets. To win the game means to have a specified positive integer m as the result. Klawe proved that when $p=\frac{1}{2}$, an (unique in some sense) optimal strategy is to bet 1 every time until m is reached. This strategy is denoted by $N(n,m)$ and is known as the bold strategy. She also showed that when $p>\frac{1}{8}(1+\sqrt{17})\simeq 0.64$, $N(6,1)$ is not optimal, and conjectured that when $p<\frac{1}{2}$, $N(n,m)$ is always optimal for any positive integers n and m. Heath [1] considered a similar problem in which winning is defined as to reach either m or $-m$. He showed that when $p=\frac{1}{2}$, the bold strategy is optimal in this setting. Sudderth and Weerasinghe [3] proved that in the continuous time version of this problem, the bold strategy is also optimal.

The subject of this paper is the game when $p\ne\frac{1}{2}$. In the first half of this paper we shall give a proof for Klawe's conjecture. In the second half we show that if $p>\frac{1}{2}$, then for some (hence, infinitely many) sufficiently large n, $N(n,m)$ is not optimal. We adopt many notations from Klawe [2] and the method used in the proof for the case $p<\frac{1}{2}$ is along the same line as Klawe's proof for the case $p=\frac{1}{2}$.

Let x be a nonnegative real number, and consider the set $\mathscr{S}_n$ of all n-bet strategies for the game. For each $S\in\mathscr{S}_n$, let $P(S,x)$ be the probability to obtain x as the result with the strategy S. We define the function

$$f_n(x)=\sup\{P(S,x)\colon S\in\mathscr{S}_n\}.$$

It is easy to see that $f_n(x)$ is nonincreasing on $[0,\infty)$ and that

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$$f_{n+1}(x) = \sup\{pf_n(x-\alpha) + (1-p)f_n(x+\alpha)\colon 0 \le \alpha \le \min\{1,x\}\}. \tag{1.1}$$

For a proof of these facts, see [2, p. 106]. Also, we have $f_n(0)=1$ for every n.

A nonnegative integer r is called an n-point if $r=0$ or $n-r$ is even. For each $x\ge 0$ we define $r(n,x)$ to be the nearest n-point to x. If x is the midpoint of two consecutive n-points, let $r(n,x)$ be the smaller one. Our definition for $r(n,x)$ differs from Klawe's when n is odd and $\frac{1}{2}<x<1$. By our definition $r(n,x)=1$ in this case but it is 0 by hers. We define by induction a sequence of functions $g_n(x)$ on $[0,\infty)$ as follows. Let $g_0(0)=1$ and $g_0(x)\equiv 0$ on $(0,\infty)$. For each $n\ge 0$, define

$$g_{n+1}(x) = pg_n(r(n,x)) + (1-p)g_n(2x - r(n,x)) \tag{1.2}$$

if $x\ge r(n,x)$, and otherwise

$$g_{n+1}(x) = pg_n(2x - r(n,x)) + (1-p)g_n(r(n,x)). \tag{1.2$'$}$$

It is easy to show by induction that $g_n(x)\le f_n(x)$. In fact $g_n(x)$ is the probability function of the n-bet game with a strategy $S(n,x)$ defined as follows. For each $1\le k\le n$, let c_k denote the result after the first $k-1$ bets. Then $|x-c_k-r(n-k, x-c_k)|$ is bet on the kth bet. (This strategy is slightly different from the one defined in [2, p. 114], but they have the same winning probability.) It will be proved that for each positive integer m, $g_n(m)$ is also the winning probability of the strategy $N(n,m)$.

In the rest of this section we give some basic difference properties of the values of $g_n(x)$ on integral points. In Section 2 we generalize these properties to arbitrary numbers and show that if $p\le\frac{1}{2}$, then $g_n(x)=f_n(x)$ for any $n\ge 0$ and $x\ge 0$. Thus, $S(n,x)$ is an optimal strategy to obtain x and $N(n,m)$ is optimal to reach m. In Section 3 we prove that if $p>\frac{1}{2}$, then for any positive integer m, there exists an integer n (depending on p and m) such that $g_k(m)<f_k(m)$ whenever $k>n$ and $k-m$ is odd.

It is easy to see that $\{g_n\}$ is an increasing sequence of decreasing left-continuous step functions and every point of discontinuity is a dyadic rational.

Proposition 1.1. *If $m\ge 1$ is an n-point and $0\le a\le 1$, and if $m\le x\le m+1-a$, then*

$$p[g_n(x-1) - g_n(x-1+a)] = (1-p)[g_n(x) - g_n(x+a)]. \tag{1.3}$$

Proof. We prove only for $x=m$. Then (1.3) follows easily. It is certainly true if $n=0$. If $n>0$, then by (1.2),

$$g_n(m-1) - g_n(m-1+a) = (1-p)[g_{n-1}(m-1) - g_{n-1}(m-1+2a)],$$

and

$$g_n(m) - g_n(m+a) = p[g_{n-1}(m-1) - g_{n-1}(m-1+2a)]. \qquad \square$$

We see from Proposition 1.1 that for any $0\le x\le 1$,

$$g_{n+1}(x) = pg_n(0) + (1-p)g_n(2x).$$

Sometimes it is more convenient to use this as the definition of $g_{n+1}(x)$. Using this as the definition we see that $g_n(x)$ is the probability function of Klawe's strategy $S(n,x)$.

Corollary 1.2. *If $n \geq 0$ and $m \geq 1$, then*

$$p[g_n(m-1)-g_n(m)] \geq (1-p)[g_n(m)-g_n(m+1)]. \tag{1.4}$$

Proof. We may suppose that $n>0$ and m is not an n-point. Then it follows from Proposition 1.1 and the facts that $g_n(m)=g_{n-1}(m)$, $g_n(m-1) \geq g_{n-1}(m-1)$, and $g_n(m+1) \geq g_{n-1}(m+1)$. □

Proposition 1.3. *If $m \geq 1$ and $n \geq 0$, then*

$$g_{n+1}(m) = pg_n(m-1)+(1-p)g_n(m+1). \tag{1.5}$$

Proof. If m is not an n-point, this is just the definition (1.2). Now let m be an n-point and suppose $n>0$. Then

$$\begin{aligned} g_{n+1}(m) &= g_n(m) \\ &= pg_{n-1}(m-1)+(1-p)g_{n-1}(m+1) \\ &= pg_n(m-1)+(1-p)g_n(m+1), \end{aligned}$$

all by the definitions of g_n and g_{n+1}. □

We see from Proposition 1.3 that $g_n(m)$ is the probability to reach m with the strategy $N(n,m)$.

2. Optimality of $S(n,x)$: $p<\frac{1}{2}$

In this section we prove that if $p<\frac{1}{2}$, then $S(n,x)$ is an optimal n-bet strategy to reach x as the result of the game. We prove at first some difference inequalities which are generalizations of Proposition 1.1 and Corollary 1.2. We assume that $p<\frac{1}{2}$ throughout this section.

Proposition 2.1. *If m is an n-point and $m \leq m+a \leq x \leq x+a \leq m+2$, then*

$$p[g_n(m)-g_n(m+a)] \geq (1-p)[g_n(x)-g_n(x+a)]. \tag{2.1(n)}$$

Corollary 2.2. *If m is an n-point and $m \leq x \leq x+a \leq m+2$, then*

$$g_n(m)-g_n(m+a) \geq g_n(x)-g_n(x+a). \tag{2.2(n)}$$

Corollary 2.3. *Let n, m, k be nonnegative integers and $a \geq 0$. Then*

$$p^k[g_n(m)-g_n(m+a)] \geq (1-p)^k[g_n(m+k)-g_n(m+k+a)]. \tag{2.3(n)}$$

Proposition 2.4. *If m is an n-point and* $\max\{m-2,0\}\le x-a\le x\le m-a\le m$, *then*

$$(1-p)[g_n(m-a)-g_n(m)]\le p[g_n(x-a)-g_n(x)]. \tag{2.4(n)}$$

Corollary 2.5. *If m is an n-point and* $\max\{m-2,0\}\le x-a\le x\le m$, *then*

$$g_n(m-a)-g_n(m)\le g_n(x-a)-g_n(x). \tag{2.5(n)}$$

Corollary 2.6. *Let n, m, k be nonnegative integers and $0\le a\le m$. Then*

$$p^k[g_n(m-a)-g_n(m)]\ge(1-p)^k[g_n(m+k-a)-g_n(m+k)]. \tag{2.6(n)}$$

These inequalities will be proved simultaneously by induction on n. It is trivial to check that they all hold for $n=0$. Now we assume that they are all true for $n=j$ and we prove them for $n=j+1$.

Proof of (2.1($j+1$)). Now we suppose that m is a $(j+1)$-point and $m\le m+a\le x\le x+a\le m+2$. The goal is to prove

$$p[g_{j+1}(m)-g_{j+1}(m+a)]\ge(1-p)[g_{j+1}(x)-g_{j+1}(x+a)]. \tag{2.1($j+1$)}$$

Assume first that $m\ge 1$. Then, by the definition (1.2),

$$g_{j+1}(m)-g_{j+1}(m+a)=p[g_j(m-1)-g_j(m-1+2a)].$$

(Notice that $a\le 1$.) If $x+a\le m+1$, then similarly

$$g_{j+1}(x)-g_{j+1}(x+a)=p[g_j(2x-m-1)-g_j(2x+2a-m-1)].$$

The inequality (2.1($j+1$)) follows from (2.1(j)).

If $x\ge m+1$, then by (2.2(j)) and (2.3(j)),

$$\begin{aligned}g_{j+1}(x)-g_{j+1}(x+a)&=(1-p)[g_j(2x-m-1)-g_j(2x+2a-m-1)]\\&\le(1-p)[g_j(m+1)-g_j(m+1+2a)]\\&\le p^2(1-p)^{-1}[g_j(m-1)-g_j(m-1+2a)].\end{aligned}$$

If $x<m+1<x+a$, then

$$\begin{aligned}g_{j+1}(x)-g_{j+1}(x+a)&=p[g_j(2x-m-1)-g_j(m+1)]\\&\quad+(1-p)[g_j(m+1)-g_j(2x+2a-m-1)]\\&\le p^2(1-p)^{-1}[g_j(2x+2a-m-3)-g_j(m-1+2a)]\\&\quad+p^2(1-p)^{-1}[g_j(m-1)-g_j(2x+2a-m-3)]\\&=p^2(1-p)^{-1}[g_j(m-1)-g_j(m-1+2a)],\end{aligned}$$

by (2.4(j)) and (2.3(j)), since $m-1+2a\le 2x-m-1$.

Now suppose $m=0$. We assume at first that 1 is a $(j+1)$-point. In the cases

$x+a\le 1$ and $x\ge 1$, (2.1($j+1$)) can be translated via (1.3) to the interval $[1,2]$, to be reduced to the case $m=1$. If $x<1<x+a$, (1.3) shows that we need only to prove

$$p[g_{j+1}(x+a-1)-g_{j+1}(a)]\ge(1-p)[g_{j+1}(x)-g_{j+1}(1)].$$

By (1.2), this is equivalent to

$$p[g_j(2x+2a-2)-g_j(2a)]\ge(1-p)[g_j(2x)-g_j(2)],$$

which is (2.4(j)) for $m=2$. Assume that 1 is a j-point. Again, it is trivial when $x+a\le 1$. If $x\ge 1$, then

$$\begin{aligned} g_{j+1}(x)-g_{j+1}(x+a) &= (1-p)[g_j(2x-1)-g_j(2x+2a-1)]\\ &\le (1-p)[g_j(1)-g_j(1+2a)]\\ &\le p[g_j(0)-g_j(2a)]\\ &= p(1-p)^{-1}[g_{j+1}(0)-g_{j+1}(a)], \end{aligned}$$

by (2.2(j)) and (2.3(j)). If $x<1<x+a$, then since $x>\frac{1}{2}$,

$$\begin{aligned} &g_{j+1}(x)-g_{j+1}(x+a)\\ &\quad = pg_j(2x-1)+(1-p)g_j(1)-pg_j(1)-(1-p)g_j(2x+2a-1)\\ &\quad \le p[g_j(2x-1)-g_j(1)+g_j(0)-g_j(2x+2a-2)]\\ &\quad = p[g_j(0)-g_j(2a)]+p[g_j(2x-1)-g_j(1)-g_j(2x+2a-2)+g_j(2a)]\\ &\quad \le p(1-p)^{-1}[g_{j+1}(0)-g_{j+1}(a)], \end{aligned}$$

by (2.3(j)) and the fact

$$g_j(2a)-g_j(1)\le g_j(2x+2a-2)-g_j(2x-1),$$

which is shown by (2.5(j)) for $2a<1$ and by (1.3) and (2.1(j)) for $2a>1$. □

Proof of (2.2($j+1$)). Suppose $m\le x\le x+a\le m+2$. Then

$$g_{j+1}(m)-g_{j+1}(m+a)\ge g_{j+1}(x)-g_{j+1}(x+a) \tag{2.2($j+1$)}$$

follows immediately from (2.1($j+1$)) for $m+a\le x$. When $m+a>x$, (2.1($j+1$)) implies

$$g_{j+1}(m)-g_{j+1}(x)\ge g_{j+1}(m+a)-g_{j+1}(x+a),$$

which is equivalent to (2.2($j+1$)). □

Proof of (2.3($j+1$)). Here we have that m and k are just nonnegative integers. In proving

$$\begin{aligned} &p^k[g_{j+1}(m)-g_{j+1}(m+a)]\\ &\quad \ge (1-p)^k[g_{j+1}(m+k)-g_{j+1}(m+k+a)], \end{aligned} \tag{2.3($j+1$)}$$

we may assume $0 \le a \le 1$, since the general case can be written as a finite sum. For this special case we need only to apply (1.3) and (2.1($j+1$)) alternatively for k times. $\square$

Proof of (2.4($j+1$)). Let m be a $(j+1)$-point and $\max\{m-2, 0\} \le x-a \le x \le m-a \le m$. We need to show

$$(1-p)[g_{j+1}(m-a)-g_{j+1}(m)] \le p[g_{j+1}(x-a)-g_{j+1}(x)]. \qquad (2.4(j+1))$$

The general case that $m \ge 3$ and $m=1$ is symmetric with (2.1($j+1$)) and can be proved similarly. So, we assume $m=2$. Then, we have

$$g_{j+1}(2-a)-g_{j+1}(2) = (1-p)[g_j(3-2a)-g_j(3)].$$

It is routine when $x-a \ge 1$. If $x \le 1$, then

$$\begin{aligned} g_{j+1}(x-a)-g_{j+1}(x) &= (1-p)[g_j(2x-2a)-g_j(2x)] \\ &\ge (1-p)[g_j(2-2a)-g_j(2)] \\ &\ge p^{-1}(1-p)^2[g_j(3-2a)-g_j(3)]. \end{aligned}$$

Here we need (2.6(j)) and the inequality

$$g_j(2x-2a)-g_j(2x) \ge g_j(2-2a)-g_j(2).$$

To show this we assume $2x \ge 2-2a$. Then by (1.3) we see it is equivalent, according as $2x-2a \ge 1$, $2x-2a < 1 \le 2x$, or $2x < 1$, to,

$$g_j(2x-2a-1)-g_j(2x-1) \ge g_j(1-2a)-g_j(1),$$

$$\begin{aligned} &g_j(0)-g_j(2x-1)+p^{-1}(1-p)[g_j(2x-2a)-g_j(1)] \\ &\quad \ge g_j(1-2a)-g_j(2x-2a)+g_j(2x-2a)-g_j(1), \end{aligned}$$

or

$$(1-p)[g_j(2x-2a)-g_j(2x)] \ge p[g_j(1-2a)-g_j(1)],$$

and each of which can be obtained from (2.2(j)) or (2.5(j)). If $x-a<1<x$, then

$$\begin{aligned} g_{j+1}(x-a)-g_{j+1}(x) &= pg_j(0)+(1-p)g_j(2x-2a)-pg_j(1)-(1-p)g_j(2x-1) \\ &= (1-p)[g_j(1)-g_j(2)+g_j(2x-2a)-g_j(2x-1)] \\ &= (1-p)[(g_j(1)-g_j(2x-1))-(g_j(2-2a)-g_j(2x-2a))] \\ &\quad +(1-p)[g_j(2-2a)-g_j(2)] \\ &\ge p[g_j(0)-g_j(2x-2)]-(1-p)[g_j(2-2a)-g_j(2x-2a)] \\ &\quad +p^{-1}(1-p)^2[g_j(3-2a)-g_j(3)] \\ &\ge p^{-1}(1-p)^2[g_j(3-2a)-g_j(3)], \end{aligned}$$

by (1.3), (2.6(j)), and (2.1(j)), since $2x-2<2-2a$. $\square$

Symmetrically we can prove $(2.5(j+1))$ and $(2.6(j+1))$, and this completes the proof.

Proposition 2.7. *Suppose m is not an n-point and $0 \leq a \leq 1$. If $m-1+a \leq x \leq m+1-a$, then*

$$g_n(m-a)-g_n(m) \leq g_n(x-a)-g_n(x), \tag{2.7(n)}$$

and

$$g_n(m)-g_n(m+a) \geq g_n(x)-g_n(x+a). \tag{2.8(n)}$$

Proof. We still proceed by induction on n. It is straightforward to check $(2.7(0))$ and $(2.8(0))$. Suppose $(2.7(j))$ and $(2.8(j))$ are true. We prove $(2.7(j+1))$ now. If $x \leq m$ and $m \geq 2$ we translate the problem to the interval $[m-2, m-1]$ via (1.3) and it becomes

$$g_{j+1}(m-1-a)-g_{j+1}(m-1) \leq g_{j+1}(x-1-a)-g_{j+1}(x-1).$$

Then it follows from (2.2). If $x-a<m<x$, then the inequality becomes

$$g_{j+1}(m)-g_{j+1}(x) \geq g_{j+1}(m-a)-g_{j+1}(x-a).$$

We note that this inequality can be reduced to

$$g_j(m-1)-g_j(2x-m-1) \geq g_j(m-2a)-g_j(2x-m-2a)$$

via (1.2) and (1.3) for $m \geq 2$, and to

$$p[g_j(0)-g_j(2x-2)] \geq (1-p)[g_j(2-2a)-g_j(2x-2a)]$$

for $m=1$. Thus $(2.7(j+1))$ follows from $(2.8(j))$ and (2.1). Suppose that $x-a \geq m$. In this case we have $a \leq \frac{1}{2}$. By (1.3) and $(2.7(j))$,

$$\begin{aligned} g_{j+1}(m-a)-g_{j+1}(m) &= p[g_j(m-2a)-g_j(m)] \\ &= (1-p)[g_j(m+1-2a)-g_j(m+1)] \\ &\leq (1-p)[g_j(2x-2a-m)-g_j(2x-m)] \\ &= g_{j+1}(x-a)-g_{j+1}(x). \end{aligned}$$

If $m=1$ and $x \leq 1$, then

$$g_{j+1}(1-a)-g_{j+1}(1) = (1-p)[g_j(2-2a)-g_j(2)]$$

and

$$g_{j+1}(x-a)-g_{j+1}(x) = (1-p)[g_j(2x-2a)-g_j(2x)].$$

The inequality $(2.7(j+1))$ follows from (1.3) and (2.2) or (2.5) as we did in the proof of $(2.4(j+1))$.

In the cases $x \geq m$, or $x<m<x+a$, or $x+a \leq m$ and $m \geq 2$, $(2.8(j+1))$ can be proved similarly. We assume $m=1$ and $x+a \leq 1$. Then by (1.3) and (2.1),

$$\begin{aligned} g_{j+1}(1)-g_{j+1}(1+a) &= (1-p)[g_j(1)-g_j(1+2a)] \\ &= p[g_j(0)-g_j(2a)] \\ &\geq (1-p)[g_j(2x)-g_j(2x+2a)] \\ &= g_{j+1}(x)-g_{j+1}(x+a), \end{aligned}$$

since $2a \leq 2x$. □

Now we are ready to prove the main result of this section that $f_n(x)=g_n(x)$ for any $n \geq 0$ and $x \geq 0$. Since $g_0(x) \equiv f_0(x)$, it is sufficient to show that

$$g_{n+1}(x) = \sup\{pg_n(x-\alpha)+(1-p)g_n(x+\alpha)\colon 0 \leq \alpha \leq \min\{1,x\}\},$$

and this is proved in the next theorem.

Theorem 2.8. *Let $m=r(n,x)$ and $0 \leq \alpha \leq \min\{1,x\}$. Then,*

$$pg_n(m)+(1-p)g_n(2x-m) \geq pg_n(x-\alpha)+(1-p)g_n(x+\alpha) \tag{2.9}$$

when $m \leq x$, and

$$pg_n(2x-m)+(1-p)g_n(m) \geq pg_n(x-\alpha)+(1-p)g_n(x+\alpha) \tag{2.10}$$

when $x<m$.

Proof. We prove (2.9) first. If $\alpha \leq x-m$, then by (2.1(n)),

$$p[g_n(m)-g_n(x-\alpha)] \geq (1-p)[g_n(x+\alpha)-g_n(2x-m)].$$

If $\alpha > x-m$, then by (1.3) and (2.7(n)),

$$\begin{aligned} p[g_n(x-\alpha)-g_n(m)] &= (1-p)[g_n(x-\alpha+1)-g_n(m+1)] \\ &\leq (1-p)[g_n(2x-m)-g_n(x+\alpha)]. \end{aligned}$$

Now we prove (2.10). If $\alpha \leq m-x$, then by (2.4(n)),

$$(1-p)[g_n(x+\alpha)-g_n(m)] \leq p[g_n(2x-m)-g_n(x-\alpha)].$$

If $\alpha > m-x$, then by (1.3) and (2.8(n)) (when $m>1$) or (2.2(n)) (when $m=1$),

$$\begin{aligned} (1-p)[g_n(m)-g_n(x+\alpha)] &= p[g_n(m-1)-g_n(x+\alpha-1)] \\ &\geq p[g_n(x-\alpha)-g_n(2x-m)]. \end{aligned}$$ □

The next theorem shows that even with this strategy, the probability to win the game is still small when x is large. When the game is fair ($p=\frac{1}{2}$), intuition tells us that any value $x>0$ can be reached if there is no limit for the number of bets. But, if $p<\frac{1}{2}$, then the probability that x can be obtained with arbitrarily many bets decreases to 0 as x tends to ∞. Let $g(x)$ be the probability that x is obtained with unlimited number of bets. Obviously, $g(x)$ is the limit of the increasing sequence of functions $g_n(x)$.

Theorem 2.9. *Let k be a positive integer. Then*

$$g(k) = \left(\frac{p}{1-p}\right)^k.$$

Proof. By Corollary 2.3, for any n, k, and a,

$$g_n(k) - g_n(k+a) \le \left(\frac{p}{1-p}\right)^k.$$

Since $g_n(x) \to 0$ as $x \to \infty$, we have $g_n(k) \le (p/(1-p))^k$. Thus, we also have $g(k) \le (p/(1-p))^k$, and hence, $g(x) \to 0$ as $x \to \infty$. By Proposition 1.1,

$$p[g(k) - g(k+1)] = (1-p)[g(k+1) - g(k+2)].$$

It follows that $g(k) = (p/(1-p))^k$. □

Remark. All results we obtained in this section hold for $p = \frac{1}{2}$.

3. Nonoptimality of $S(n,x)$: $p > \frac{1}{2}$

In this section we prove that for $p > \frac{1}{2}$, if n is odd and sufficiently large, then, $pg_n(\frac{1}{2}) + (1-p)g_n(\frac{3}{2}) > g_{n+1}(1)$. Thus, $f_{n+1}(1) > g_{n+1}(1)$ and $S(n+1, 1)$ is not optimal. This implies that $f_{n+m}(m) > g_{n+m}(m)$ and $S(n+m, m)$ is not optimal.

Lemma 3.1. *Suppose $n \ge m-1$ and m is not an n-point. Then*

$$\frac{g_n(m) - g_n(m+1)}{g_n(m-1) - g_n(m)} \le \frac{g_{n+2}(m) - g_{n+2}(m+1)}{g_{n+2}(m-1) - g_{n+2}(m)}.$$

Proof. We again prove by induction on n. The first step follows from $g_{m-1}(m) = g_{m-1}(m+1) = 0$ but $g_{m-1}(m-1) = p^{m-1}$. Suppose $n \ge m+1$. If $m > 1$, then by Propositions 1.3 and 1.1 and the inductive hypothesis,

$$\begin{aligned}
&\frac{g_{n+2}(m) - g_{n+2}(m+1)}{g_{n+2}(m-1) - g_{n+2}(m)} \\
&= \frac{p[g_{n+1}(m-1) - g_{n+1}(m)] + (1-p)[g_{n+1}(m+1) - g_{n+1}(m+2)]}{p[g_{n+1}(m-2) - g_{n+1}(m-1)] + (1-p)[g_{n+1}(m) - g_{n+1}(m+1)]} \\
&= \frac{1 + [g_{n+1}(m+1) - g_{n+1}(m+2)]/[g_{n+1}(m) - g_{n+1}(m+1)]}{[g_{n+1}(m-2) - g_{n+1}(m-1)]/[g_{n+1}(m-1) - g_{n+1}(m)] + 1} \\
&\ge \frac{1 + [g_{n-1}(m+1) - g_{n-1}(m+2)]/[g_{n-1}(m) - g_{n-1}(m+1)]}{[g_{n-1}(m-2) - g_{n-1}(m-1)]/[g_{n-1}(m-1) - g_{n-1}(m)] + 1} \\
&= \frac{g_n(m) - g_n(m+1)}{g_n(m-1) - g_n(m)}.
\end{aligned}$$

If $m=1$, then by Proposition 1.1 and the inductive hypothesis,

$$\begin{aligned}
&\frac{g_{n+2}(1)-g_{n+2}(2)}{1-g_{n+2}(1)}\\
&\quad=\frac{p[1-g_{n+1}(1)]+(1-p)[g_{n+1}(2)-g_{n+1}(3)]}{(1-p)[1-g_{n+1}(2)]}\\
&\quad=\frac{(1-p)[g_{n+1}(1)-g_{n+1}(2)]+(1-p)[g_{n+1}(2)-g_{n+1}(3)]}{(1-p)p^{-1}[g_{n+1}(1)-g_{n+1}(2)]}\\
&\quad=p\left[1+\frac{g_{n+1}(2)-g_{n+1}(3)}{g_{n+1}(1)-g_{n+1}(2)}\right]\\
&\quad\geq p\left[1+\frac{g_{n-1}(2)-g_{n-1}(3)}{g_{n-1}(1)-g_{n-1}(2)}\right]=\frac{g_n(1)-g_n(2)}{1-g_n(1)}. \qquad \square
\end{aligned}$$

Theorem 3.2. *There exists an odd integer n such that*

$$pg_n(\tfrac{1}{2})+(1-p)g_n(\tfrac{3}{2})>g_{n+1}(1).$$

Proof. From Lemma 3.1 we see that the limit

$$\lim_{k\to\infty}\frac{g_{m+2k+1}(m)-g_{m+2k+1}(m+1)}{g_{m+2k+1}(m-1)-g_{m+2k+1}(m)}=\lambda_m$$

exists as a positive number or $+\infty$ for each $m\geq 1$. If the theorem is not true, then for every odd integer n,

$$\begin{aligned}
pg_{n-1}(0)+(1-p)g_{n-1}(2) &= g_n(1)=g_{n+1}(1)\\
&\geq pg_n(\tfrac{1}{2})+(1-p)g_n(\tfrac{3}{2})\\
&=p^2g_{n-1}(0)+2p(1-p)g_{n-1}(1)+(1-p)^2g_{n-1}(2).
\end{aligned}$$

This implies that

$$g_{n-1}(0)+g_{n-1}(2)\geq 2g_{n-1}(1),$$

and hence $\lambda_1\leq 1$. From the proof of Lemma 3.1 we see that

$$\lambda_1=p(1+\lambda_2)\quad\text{and}\quad \lambda_m(\lambda_{m-1}^{-1}+1)=1+\lambda_{m+1},\quad m>1.$$

From this it is easy to prove by induction that $\lambda_m\leq\lambda_{m-1}$ for all $m\geq 1$, and hence $\lambda_1\leq 1$ implies that $\lambda_m\leq 1$ for any $m\geq 1$. Choose a large even integer k. Denote by

$$\alpha_m=g_k(m-1)-g_k(m),$$

$$\delta_m=g_{k+2}(m)-g_k(m),$$

$$\theta_m=\frac{g_k(m)-g_k(m+1)}{g_k(m-1)-g_k(m)},$$

for $1 \le m \le k+1$. Notice the following relations among these constants:

$$\theta_m = \frac{p}{1-p}, \qquad m \text{ even},$$

$$\frac{\alpha_{m+1}}{\alpha_m} = \theta_m,$$

$$\delta_m = p\alpha_m - (1-p)\alpha_{m+1}, \qquad m \text{ odd},$$

$$\delta_m = p\delta_{m-1} + (1-p)\delta_{m+1}, \qquad m \text{ even},$$

$$\frac{\delta_m - \delta_{m+1} + \alpha_{m+1}}{\delta_{m-1} - \delta_m + \alpha_m} \le \lambda_m, \qquad m \text{ odd},$$

$$\frac{\delta_m - \delta_{m+1}}{\delta_{m-1} - \delta_m} = \frac{p}{1-p}, \qquad m \text{ even}.$$

The last two imply that

$$\delta_{m+1} = \frac{\delta_m - p\delta_{m-1}}{1-p}, \qquad m \text{ even},$$

$$\delta_{m+1} \ge \delta_m(1+\lambda_m) - \lambda_m\delta_{m-1} - (\lambda_m - \theta_m)\alpha_m, \quad m \text{ odd}, \tag{3.1}$$

and we see that the ratio of the two sides in the inequality (3.1) tends to 1 as $k \to \infty$, for each m. Notice that $\theta_m \to \lambda_m$ as $k \to \infty$, for each odd m. Thus for a given integer l, we can choose k large enough such that the term $(\lambda_m - \theta_m)\alpha_m$ is small relative to δ_1, for $m = 1, 3, \ldots, 2l-1$, since $\delta_1 = (p - (1-p)\theta_1)\alpha_1$ and $\alpha_{m+1} = \alpha_1\theta_1 \cdots \theta_m \le \alpha_1(p/(1-p))^{m/2}$. Now by induction we can show that the first $2l$ terms of $\{\delta_m\}$ are increasing and for each odd $m < 2l$, we can write

$$\delta_{m+1} = [\delta_m(1+\lambda_m) - \lambda_m\delta_{m-1}](1+\varepsilon_m),$$

where $\varepsilon_m \to 0$ as $k \to \infty$. Inductively, we calculate that

$$\delta_1 = [p - (1-p)\lambda_1]\alpha_1(1+\varepsilon_1'),$$

$$\delta_2 = (1+\lambda_1)\delta_1(1+\varepsilon_2'),$$

$$\delta_3 = \left(1 + \frac{\lambda_1}{1-p}\right)\delta_1(1+\varepsilon_3'),$$

$$\delta_4 = \left(1 + \frac{1}{1-p}\lambda_1 + \frac{p}{1-p}\lambda_1\lambda_3\right)\delta_1(1+\varepsilon_4'),$$

$$\vdots$$

$$\delta_{2l-1} = \left(1 + \frac{1}{1-p}\lambda_1 + \frac{p}{(1-p)^2}\lambda_1\lambda_3 + \cdots + \frac{p^{l-2}}{(1-p)^{l-1}}\lambda_1\lambda_3\cdots\lambda_{2l-3}\right)\delta_1(1+\varepsilon_{2l-1}'),$$

$$\delta_{2l} = \left(1 + \frac{1}{1-p}\lambda_1 + \frac{p}{(1-p)^2}\lambda_1\lambda_3 + \cdots + \frac{p^{l-2}}{(1-p)^{l-1}}\lambda_1\lambda_3\cdots\lambda_{2l-3} + \frac{p^{l-1}}{(1-p)^{l-1}}\lambda_1\lambda_3\cdots\lambda_{2l-1}\right)\delta_1(1+\varepsilon'_{2l}),$$

and $\varepsilon'_j \to 0$ as $k\to\infty$, for $j=1,2,\ldots,2l$. We take δ_{2l-1}; then

$$\begin{aligned}
&\left(1 + \frac{1}{1-p}\lambda_1 + \frac{p}{(1-p)^2}\lambda_1\lambda_3 + \cdots + \frac{p^{l-2}}{(1-p)^{l-1}}\lambda_1\lambda_3\cdots\lambda_{2l-3}\right)(p-(1-p)\lambda_1)\alpha_1(1+\varepsilon)\\
&= \delta_{2l-1} = p\alpha_{2l-1} - (1-p)\alpha_{2l} \le p\alpha_{2l-1}\\
&= p\alpha_1\theta_1\theta_2\cdots\theta_{2l-2} \le p\alpha_1\left(\frac{p}{1-p}\right)^{l-1}\lambda_1\lambda_3\cdots\lambda_{2l-3};
\end{aligned}$$

or

$$\begin{aligned}
p^l &\ge (p-(1-p)\lambda_1)\left[\frac{(1-p)^{l-1}}{\lambda_1\lambda_3\cdots\lambda_{2l-3}} + \frac{(1-p)^{l-2}}{\lambda_3\cdots\lambda_{2l-3}} + \frac{p(1-p)^{l-3}}{\lambda_5\cdots\lambda_{2l-3}} + \cdots + p^{l-2}\right]\\
&\ge (2p-1)[(1-p)^{l-1} + (1-p)^{l-2} + p(1-p)^{l-3} + \cdots + p^{l-2}]\\
&= (2p-1)\left[(1-p)^{l-1} + \frac{p^{l-1}-(1-p)^{l-1}}{2p-1}\right]\\
&= [p^{l-1} - 2(1-p)^l].
\end{aligned}$$

This is a contradiction since $p>\frac{1}{2}$, and it completes the proof. □

Remark 1. From Lemma 3.1 we see that if Theorem 3.2 is satisfied for some odd integer n, then it is satisfied for every odd integer greater than n. Therefore, $f_{n+m}(m) > g_{n+m}(m)$ for all m and all large odd n.

Remark 2. Theorem 3.2 does not tell us how large the integer could be. By some computing work we find a sequence $\{p_n\}$ of values of the probability p such that $p>p_n$ implies that $g_{2n}(0)+g_{2n}(2)<2g_{2n}(1)$, which implies the $g_{2n+2}(1) < pg_{2n+1}(\frac{1}{2}) + (1-p)g_{2n+1}(\frac{3}{2})$. Here are some values of p_n:

$$p_{10} = 0.586616\ldots,$$

$$p_{20} = 0.532870\ldots,$$

$$p_{50} = 0.512787\ldots,$$

$$p_{100} = 0.505954\ldots.$$

We see that p_n converges to $\frac{1}{2}$ slowly.

Remark 3. The author is grateful to the referees for their many suggestions which make the work more complete and easier to read.

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